Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

NOTE ON THE DUAL OF A FOCAL PROPERTY OF THE INSCRIBED ELLIPSE.

By R. E. Allardice.

The focal property of the inscribed ellipse referred to in the title is the one to which the well-known theorem given by Steiner in an article entitled *Développement d'une série de théorèmes relatifs aux sections coniques,*[*] relates, namely, that any two points that are isogonal conjugates[†] with reference to a triangle are the foci of an ellipse inscribed in the triangle. As the focal properties of a conic are metrical, we cannot be certain of recognizing all possible aspects of their duals. It is, however, not difficult to see that we may find a dual of the theorem that the focal radii are equally inclined to the tangent, in the theorem that the intercept on the tangent to a hyperbola made by the asymptotes is bisected by the point of contact.[‡] We may also (especially if we consider the analogue in spherical geometry) regard the theorem that the tangent to a hyperbola forms with the asymptotes a triangle of constant area, as a dual of the theorem that the sum of the focal radii of an ellipse is constant.

The theorem that is the subject of this note is as follows:

Any two straight lines that are isotomic conjugates with respect to a triangle are the asymptotes of a hyperbola circumscribing the triangle.

---

[*] *Annales de Mathématiques*, vol. 19, pp. 37-64; *Gesammelte Werke*, vol. 1, pp. 191-210.

[†] Isogonal conjugates are defined thus: Two lines through the vertex of an angle are isogonal conjugates if they are harmonically separated by the two bisectors of the angle. With respect to a triangle, two points are called isogonal conjugates when the lines joining them to the three vertices of that triangle are pairs of isogonally conjugate lines in the three angles. *Cf.*, for example, Lachlan, *Modern Geometry*, pp. 55-56.

To define similarly isotomic conjugates, call the midpoint of a segment of a straight line and the infinitely distant point of that line the bisectors of the segment: then two points are isotomic conjugates in a segment of a line if they are harmonically separated by its bisectors. With respect to the three sides of a triangle, two lines are called isotomic conjugates when their points of intersection with those sides are pairs of isotomically conjugate points with respect to the segments included between pairs of vertices.

[‡] The lemma on which the duality of the figures is based is apparently the following, infrequent in text-books: *If two sides of a triangle inscribed in an ellipse pass through the foci, the pole of the third side lies on the normal at the opposite vertex of the triangle.* This pole being taken as centre of a circle, the two figures referred to are dual with respect to the circle.—[Ed.]

The proof by elementary geometry is immediate. Let *PD* and *PE* be isotomic conjugates with reference to the triangle *ABC* (Fig. 1); through *A* and *C* draw *AH* and *CM*, *AN* and *CL* parallel to *PD* and *PE*.

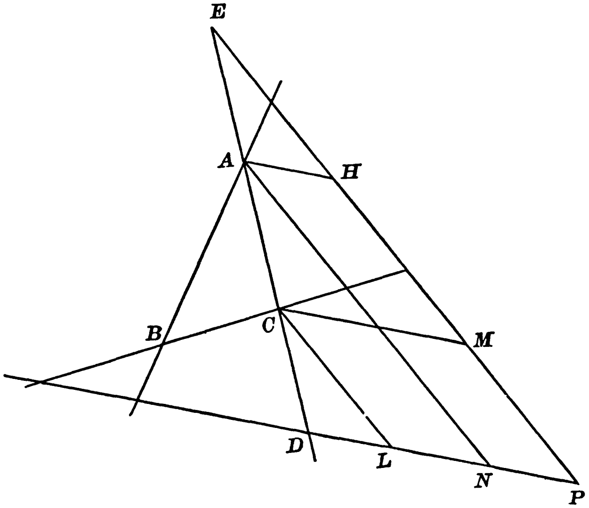

FIG. 1.

Then the triangles *AHE* and *DLC* are congruent, as are also the triangles *DNA* and *CME*. Hence the parallelograms *AP* and *CP* are equal in area; and hence a hyperbola may be constructed with *PD* and *PE* as asymptotes, to pass through *A*, *B* and *C*.

Thus if the envelope of one asymptote of a conic circumscribing a triangle be given, we can write down the equation of the envelope of the other. In particular, if the first asymptote pass through a fixed point, the second asymptote will envelope a conic inscribed in the given triangle.

I have obtained the following results: if

$$l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0$$

be the equation of a circumscribed conic, then the equations of the asymptotes are

$$a^2\alpha + [\{\Delta + 2ablm \pm (al + bm - cn)\Delta^{\frac{1}{2}}\}/2lm]\beta$$
$$+ [\{\Delta + 2acln \mp (al - bm + cn)\Delta^{\frac{1}{2}}\}/2ln]\gamma = 0,$$

where $\Delta = a^2l^2 + b^2m^2 + c^2n^2 - 2ablm - 2bcmn - 2canl.$

Further if $p\alpha + q\beta + r\gamma = 0$ be the equation of one of the asymptotes, then l, m, n, are determined by the equations

$$l/p\,(br - cq)^2 = m/q\,(cp - ar)^2 = n/r\,(aq - bp)^2.$$

It is well known that the nine-point circle is the locus of the centre of an equilateral hyperbola circumscribing a given triangle. The following simple proof, depending on the foregoing theorem, seems worth noticing.

Let ABC (Fig. 2), be any triangle, and DEF, $D'E'F'$ a pair of isotomic conjugates intersecting at right angles in P. These lines are by the former theorem the asymptotes, P the centre, of an equilateral hyperbola circumscribed to the triangle ABC. The locus of P is to be found. Let L, M, N, be the middle points of the sides, then L, M, N, are likewise the middle points of DD', EE', FF'; hence we have

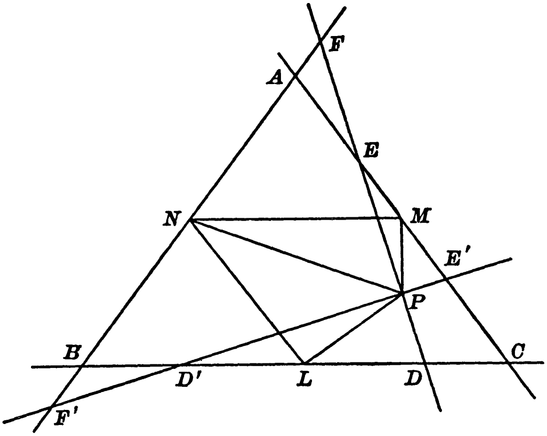

FIG. 2.

$$\angle NPM = \angle NPF + \angle MPE = \angle NFP + \angle MEP = A = \angle NLM.$$

Hence P lies on the circle passing through L, M, N, that is, on the nine-point circle.